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Stability of rigid motions and rollers in bicomponent flows of immiscible liquids

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We consider the motion of two rings of liquids with different viscosities and densities lying between concentric cylinders that rotate with the same angular velocity Ω. Gravity is neglected and interfacial tension is included. We show that rigid motions are globally stable and that the shape of the interface which separates the two fluids is determined by a minimizing problem for a potential $\mathscr{P}$ defined as the negative of the sum of the kinetic energies of two rigid motions plus the surface energy of the interface. We show that the stable interface between fluids has a constant radius when heavy fluid is outside and $J = -d^3[\![\rho]\!]\,\Omega^2/T$ is larger than four, where d is the mean radius, $[\![\rho]\!] < 0$ the density difference and T the surface tension. When J is negative the heavy fluid is inside and the interface must be corrugated. The potential of flows with heavy fluid outside is smaller, thus relatively more stable, than when light fluid is outside, whenever J is large or for any J when the volume ratio m of heavy to light fluid is greater than one. These results give partial explanation of the stability and shape of rollers of viscous oils rotating in water and the corrugation of the free surface of films coating rotating cylinders.

1. Introduction

We consider the flow of two immiscible liquids with different viscosities and densities lying between concentric cylinders, both of which rotate with the same angular velocity Ω. We neglect gravity and include interfacial tension. We study the stability of steady rigid-body rotation in which the two fluids are arranged in two rings with a given volume ratio. We show that rigid-body rotation is globally stable, and the interface shape between the two fluids is determined by a minimizing problem for a potential defined as the negative of the sum of the kinetic energies of two rigid motions plus the surface energy of the interface. We show that the interface between the two fluids has a constant radius when the heavy fluid is outside and $J = -d^3[\![\rho]\!]\,\Omega^2/T$ is larger than four. This implies that centrifuged configurations lie outside an interface of constant radius. We note that the stable rollers of oil in water observed by Joseph, Nguyen & Beavers (1984) have heavy fluid (water) outside an interface of essentially constant radius. The rollers are maintained in nearly rigid motion by the high viscosity; there is no outer cylinder, and the motion of the water is not rigid. We show that the interface on rigid motions with heavy fluid inside must

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be corrugated. Photographs of corrugated interfaces of liquid films coating cylinders rotating in air can be found in the papers of Yih (1960) and Moffatt (1977). Yih gave a linear stability analysis for films coating cylinders rotating in air, and his results are consistent with ours.

The potential of flows with heavy fluid outside is smaller, thus relatively more stable, than when light fluid is outside, when J is large or for any J when the volume ratio m of heavy to light fluid is greater than one. This is consistent with the idea that configurations with heavier fluid outside should be more stable because of the centrifugal force and that, if the inner fluid is heavier, the rigid motion should be less stable no matter what the viscosities. The stability of flows in which viscosity differences are important depend strongly on the viscosity ratio. Rigid-body rotation involves no shear. As a result, the mechanism that we call lubrication stabilization (Renardy & Joseph 1985), in which thin layers of the less viscous fluid occupy regions of high shear, is absent.

2. Equations of motion and interface conditions

Consider the flow of two immiscible liquids contained between two infinite concentric cylinders. The perturbed regions occupied by liquid 1 and 2 are denoted respectively as

$$\mathscr{V}_1(t) = \{r,\theta,x\,|\,a \leqslant r \leqslant R(x,\theta,t),\ -\infty < x < \infty,\ 0 \leqslant \theta \leqslant 2\pi\},$$

$$\mathscr{V}_2(t) = \{r,\theta,x\,|\,R(x,\theta,t) \leqslant r \leqslant b,\ -\infty < x < \infty,\ 0 \leqslant \theta \leqslant 2\pi\}.$$

The stress is given by

$$\boldsymbol{T} = -p\boldsymbol{I}+\boldsymbol{S}, \quad \boldsymbol{S} = 2\mu\mathbf{D}[\boldsymbol{u}]. \tag{2.1}$$

The equations of motion hold in each region:

$$\operatorname{div}\boldsymbol{u}_l = 0, \quad l = 1,2, \tag{2.2}$$

$$\rho_l \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_l = -\nabla\Phi_l+\operatorname{div}\boldsymbol{S}_l, \tag{2.3}$$

where $\boldsymbol{u}_l = \boldsymbol{e}_r u_l+\boldsymbol{e}_\theta v_l+\boldsymbol{e}_x w_l$, $\Phi_l = p_l+\rho_l gr\sin\theta$, $\mathbf{D}[\boldsymbol{u}] = \frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\mathrm{T}})$, ρ_1 and ρ_2 are the densities and μ_1 and μ_2 are the viscosities. In all that follows $g = 0$. The cylinders at $r = a$ and $r = b$ rotate with some constant angular frequency Ω. At the interface Σ given by

$$F(x,r,\theta,t) = r-R(x,\theta,t) = 0, \tag{2.4}$$

we have

$$u = \frac{\partial R}{\partial t}+\frac{v}{R}\frac{\partial R}{\partial \theta}+w\frac{\partial R}{\partial x}. \tag{2.5}$$

We also require that the jumps across Σ,

$$[\![\ ]\!] = (\ )_1-(\ )_2,$$

in the velocity, the shear stress and the difference between the jump in the normal stress and the surface tension force all vanish.

We are going to study spatially periodic solutions that are $(2\pi/\alpha)$-periodic in x and, of course, 2π-periodic in θ. The volume of each component fluid is prescribed by specifying a mean radius

$$d^2 = \overline{R}^2, \tag{2.6}$$

where
$$\overline{(\ )} = \frac{\alpha}{4\pi^2}⟪\ ⟫, \quad ⟪\ ⟫ \equiv \int_0^{2\pi/\alpha} \mathrm{d}x \int_0^{2\pi} (\ )\,\mathrm{d}\theta. \tag{2.7}$$

Our convention is the fluid with subscript 1 is on the inside. The fluid on the inside can be heavy or light. We are interested in two cases:

(A) the heavy fluid inside, $a \leqslant r \leqslant R_A(\theta, x, t)$,
$$[\![\rho]\!] > 0, \quad d_A^2 = \overline{R_A^2}; \tag{2.8}$$

(B) the heavy fluid is outside, $R_B(\theta, x, t) \leqslant r \leqslant b$,
$$[\![\rho]\!] < 0, \quad d_B^2 = \overline{R_B^2}. \tag{2.9}$$

The volume ratio of heavy to light fluid is
$$m_A = \frac{d_A^2 - a^2}{b^2 - d_A^2}, \quad m_B = \frac{b^2 - d_B^2}{d_B^2 - a^2}. \tag{2.10}$$

The volume of light fluid and the volume of heavy fluid is fixed, independent of whether it is inside or outside when $m_A = m_B$. Then
$$d_A^2 = \frac{m_b^2 + a^2}{1+m}, \quad d_B^2 = \frac{b^2 + m_a^2}{1+m}. \tag{2.11}$$

The jump equations at $r = R$ are
$$[\![\boldsymbol{u}]\!] = 0, \quad -([\![p]\!] + 2HT)\,\boldsymbol{n} + [\![\boldsymbol{S}]\!]\cdot\boldsymbol{n} = 0, \tag{2.12}$$

where
$$2H = \frac{RR_{\theta\theta}(1+R_x^2) + RR_{xx}(R^2+R_\theta^2) - R^2(1+R_x^2) - 2R_\theta^2 - 2RR_\theta R_x R_{x\theta}}{(R^2 + R_\theta^2 + R^2R_x^2)^{\frac{3}{2}}}. \tag{2.13}$$

$2H$ is the sum of the principal curvatures, T is the surface tension, and $\boldsymbol{n} = \boldsymbol{\nabla}F/|\boldsymbol{\nabla}F|$ and $\boldsymbol{\nabla}F = \boldsymbol{e}_r - \boldsymbol{e}_\theta R_\theta/R - \boldsymbol{e}_x R_x$.

3. Rigid rotation of two fluids

The velocity
$$\boldsymbol{u}_0 = \boldsymbol{e}_\theta \Omega r \tag{3.1}$$

and the pressure
$$p_0 = \tfrac{1}{2}\rho\Omega^2 r^2 + D, \tag{3.2}$$

where the constants (ρ, D) are (ρ_1, D_1) in $\mathscr{V}_1$ and (ρ_2, D_2) in $\mathscr{V}_2$, is a solution of (2.2) and (2.3) with $\boldsymbol{S}$ identically zero. We suppose that $R(\theta, x, t)$, periodic in x and θ, is prescribed and arbitrary. At $r = R$ we have
$$[\![\boldsymbol{u}_0]\!] = 0, \quad [\![p_0]\!] = [\![\rho]\!]\tfrac{1}{2}\Omega^2R^2 + [\![D]\!]. \tag{3.3}$$

We cannot satisfy the differential equation
$$[\![p_0]\!] + 2HT = 0, \tag{3.4}$$

expressing the jump condition for the normal component of the stress, for an arbitrary given surface $R(\theta, x, t)$. We call (3.1)–(3.3) an 'extended' rigid motion, and we prove that these motions are globally stable with the shape of the interface, and possibly the placement of heavy and light liquid determined by a new minimum principle of classical type.

4. Perturbation equations

Let $\boldsymbol{u}$, p be the velocity and pressure in the deformed domain, and let

$$\boldsymbol{u} = \boldsymbol{u}_0 + \hat{\boldsymbol{u}}, \quad p = p_0 + \hat{p},$$

where $\boldsymbol{u}_0 = \boldsymbol{e}_\theta V(r)$, and $p_0(r)$ is the associated pressure. The functions $\hat{\boldsymbol{u}}$, $\hat{p}$ and $\hat{\boldsymbol{S}}$ are perturbations. In designating components

$$\hat{\boldsymbol{u}} = (u, v, w)$$

we suppress the caret overbar. All these quantities are defined in $\mathscr{V}_1(t)$ and $\mathscr{V}_2(t)$. For the moment we leave open the possibility that $[\![V]\!] \neq 0$ (for rigid motion $[\![V]\!] = 0$). The equations governing $\hat{\boldsymbol{u}}$ and $\hat{p}$ are

$$\rho\left[\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}_0 + \boldsymbol{u}_0\cdot\nabla\hat{\boldsymbol{u}} + \hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\right] = -\nabla\hat{p} + \nabla\cdot\hat{\boldsymbol{S}}, \tag{4.1}$$

$$\nabla\cdot\hat{\boldsymbol{u}} = 0. \tag{4.2}$$

The boundary conditions are

$$\hat{\boldsymbol{u}}_1(r = a) = \hat{\boldsymbol{u}}_2(r = b) = 0. \tag{4.3a, b}$$

The interface conditions on $r = R(x, \theta, t)$ are

$$[\![u]\!] = [\![w]\!] = [\![v + V]\!] = 0, \tag{4.4}$$

$$u = \frac{\partial R}{\partial t} + \frac{v + V}{R}\frac{\partial R}{\partial \theta} + w\frac{\partial R}{\partial x}, \tag{4.5}$$

$$-[\![\hat{p}]\!]\,\boldsymbol{n} + [\![\hat{\boldsymbol{S}}]\!]\cdot\boldsymbol{n} = [\![p_0]\!]\,\boldsymbol{n} + 2HT\boldsymbol{n}. \tag{4.6}$$

5. Energy equations for nonlinear disturbances

We introduce the following notation:

$$\langle\;\rangle = \int_{\mathscr{V}} (\;)\,\mathrm{d}\mathscr{V}, \quad \langle\;\rangle_\Sigma = \int_\Sigma (\;)\,\mathrm{d}\Sigma,$$

where
$$\mathrm{d}\Sigma = R\,\mathrm{d}\theta\,\mathrm{d}x\left[1 + \frac{R_\theta^2}{R^2} + R_x^2\right]^{\frac{1}{2}} = R\,\mathrm{d}\theta\,\mathrm{d}x\,|\nabla F|,$$

and we are assuming that the disturbance flow is $(2\pi/\alpha)$-periodic in x.

To form the energy equation, we multiply (4.1) by $\boldsymbol{u}$, integrate over $\mathscr{V}_1$ and $\mathscr{V}_2$, add, and use Reynolds' transport theorem to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\rho\frac{|\hat{\boldsymbol{u}}|^2}{2}\right\rangle = \left\langle\rho\hat{\boldsymbol{u}}\cdot\frac{\mathrm{d}\hat{\boldsymbol{u}}}{\mathrm{d}t}\right\rangle$$
$$= \left\langle\rho\hat{\boldsymbol{u}}\cdot\left[\frac{\partial\hat{\boldsymbol{u}}}{\partial t} + (\hat{\boldsymbol{u}} + \boldsymbol{u}_0)\cdot\nabla\hat{\boldsymbol{u}}\right]\right\rangle$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\rho\frac{|\hat{\boldsymbol{u}}|^2}{2}\right\rangle + \langle\rho\hat{\boldsymbol{u}}\cdot\mathbf{D}[\boldsymbol{u}_0]\cdot\hat{\boldsymbol{u}}\rangle + \langle 2\mu\mathbf{D}[\hat{\boldsymbol{u}}]:\mathbf{D}[\hat{\boldsymbol{u}}]\rangle = \langle[\![-\hat{p}\boldsymbol{u}\cdot\boldsymbol{n} + \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{S}}\cdot\boldsymbol{n}]\!]\rangle_\Sigma. \tag{5.1}$$

For flow in circles with $\boldsymbol{u}_0 = \boldsymbol{e}_\theta r\Omega$ we have

$$\langle \rho \hat{\boldsymbol{u}} \cdot \mathbf{D}[\boldsymbol{u}_0] \cdot \hat{\boldsymbol{u}} \rangle = \left\langle \rho u v r \frac{\mathrm{d}\Omega}{\mathrm{d}r} \right\rangle,$$

which vanishes on rigid motions. Moreover, since $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_0$ is continuous across Σ, we have

$$\langle [\![\hat{\boldsymbol{u}} \cdot (-\hat{p}\boldsymbol{n} + \boldsymbol{S} \cdot \boldsymbol{n})]\!] \rangle_\Sigma = \langle (\boldsymbol{u} - \boldsymbol{u}_0) \cdot [\![-\hat{p}\boldsymbol{n} + \boldsymbol{S} \cdot \boldsymbol{n}]\!] \rangle_\Sigma = \langle (\boldsymbol{u} - \boldsymbol{u}_0) \cdot \boldsymbol{n} ([\![p_0]\!] + 2HT) \rangle_\Sigma, \tag{5.2}$$

where the last equality follows from (4.6). We next observe, following ideas of E. Dussan V., (see Joseph 1976, equation (96.11)) that

$$\langle 2HT\boldsymbol{u} \cdot \boldsymbol{n} \rangle_\Sigma = -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma T \, \mathrm{d}\Sigma. \tag{5.3}$$

Since T is constant, we have

$$\langle 2HT\boldsymbol{u} \cdot \boldsymbol{n} \rangle_\Sigma = -T \frac{\mathrm{d}}{\mathrm{d}t} \langle\!\langle [R^2 + R_\theta^2 + R^2 R_x^2]^{\frac{1}{2}} \rangle\!\rangle,$$

where $\langle\!\langle \ \rangle\!\rangle$ is defined in (2.7).

Moreover, using (2.5), we find that

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \frac{\boldsymbol{\nabla} F}{|\boldsymbol{\nabla} F|} = \frac{1}{|\boldsymbol{\nabla} F|} \frac{\partial R}{\partial t}.$$

Hence $$\langle [\![p_0]\!] \boldsymbol{u} \cdot \boldsymbol{n} \rangle_\Sigma = \langle\!\langle [\![p_0]\!] R \, \partial R / \partial t \rangle\!\rangle.$$

Since $[\![p_0]\!] = [\![\rho]\!] \frac{1}{2} \Omega^2 R^2 + [\![D]\!]$ is a function of R alone, there is a scalar function

$$\Phi(R) = [\![\rho]\!] \tfrac{1}{8} \Omega^2 (R^2 - d^2)^2 \tag{5.4}$$

such that $$\langle\!\langle [\![p_0]\!] R \, \partial R / \partial t \rangle\!\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle\!\langle \Phi(R) \rangle\!\rangle; \tag{5.5}$$

in deriving (5.4)–(5.5), we used (2.6) to set

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\!\langle R^2 \rangle\!\rangle = 0.$$

Finally, we show that, for rigid motions where $\boldsymbol{u}_0 \cdot \boldsymbol{n} = \Omega R_\theta / |\boldsymbol{\nabla} F|$, we have

$$\langle \boldsymbol{u}_0 \cdot \boldsymbol{n} ([\![p_0]\!] + 2HT) \rangle_\Sigma = \Omega \langle\!\langle R R_\theta ([\![p_0]\!] + 2HT) \rangle\!\rangle = 0. \tag{5.6}$$

In deriving the first equality, we use $\boldsymbol{n} = \boldsymbol{\nabla} F / |\boldsymbol{\nabla} F|$; then we note that, since $[\![p_0]\!]$ is a function of R alone, there is a 2π-periodic $m(R)$ such that

$$R_\theta R [\![p_0]\!] = \frac{\partial m(R)}{\partial \theta},$$

which vanishes on integration. We next use the expression (2.13) to write

$$R R_\theta 2H = \frac{\partial}{\partial \theta} \left[\frac{-R(1 + R_x^2)}{|\boldsymbol{\nabla} F|} \right] + \frac{\partial}{\partial x} \left[\frac{R_x R_\theta R}{|\boldsymbol{\nabla} F|} \right].$$

Since $R(\theta, x, t)$ is periodic in x and θ, the last integral in (5.6) vanishes.

Collecting all these results, we find that

$$\frac{\mathrm{d}(\mathscr{E}+\mathscr{M})}{\mathrm{d}t} = -\mathscr{D}, \tag{5.7}$$

where

$$\left.\begin{aligned}\mathscr{E} &= \langle \tfrac{1}{2}\rho|\hat{\boldsymbol{u}}|^2\rangle, \quad \mathscr{D} = \langle 2\mu \mathbf{D}[\hat{\boldsymbol{u}}]:\mathbf{D}[\hat{\boldsymbol{u}}]\rangle, \\ \mathscr{M} &= T\langle\!\langle [R^2+R_\theta^2+R^2R_x^2]^{\frac{1}{2}}\rangle\!\rangle - \tfrac{1}{8}[\![\rho]\!]\,\Omega^2\,\langle\!\langle [R^2-d^2]^2\rangle\!\rangle.\end{aligned}\right\} \tag{5.8}$$

The function $\mathscr{M}[R]$ is the variable part of a 'potential energy' $\mathscr{P}[R]$ for rigid motions. It is easily verified that

$$\mathscr{K}_1 = \tfrac{1}{8}[-\langle\!\langle a^4\rangle\!\rangle + \langle\!\langle R^4\rangle\!\rangle]\,\rho_1\Omega^2, \quad \mathscr{K}_2 = \tfrac{1}{8}[\langle\!\langle b^4\rangle\!\rangle - \langle\!\langle R^4\rangle\!\rangle]\,\rho_2\Omega^2,$$

are the kinetic energies of rigid motions in regions $\mathscr{V}_1$ and $\mathscr{V}_2$ respectively. The associated potential of these rigid motions is the negative of the sum of the kinetic energies plus the surface energy

$$\begin{aligned}\mathscr{P} &\equiv -(\mathscr{K}_1+\mathscr{K}_2) + T\int_\Sigma \mathrm{d}\Sigma \\ &= C_1 + T\langle\!\langle [R^2(1+R_x^2)+R_\theta^2]^{\frac{1}{2}}\rangle\!\rangle - \tfrac{1}{8}[\![\rho]\!]\,\langle\!\langle R^4\rangle\!\rangle\,\Omega^2 \\ &\equiv C_1 + Q = C_2 + \mathscr{M},\end{aligned} \tag{5.9}$$

where C_1 and C_2 are different constants. We may write (5.7) as

$$\frac{\mathrm{d}(\mathscr{E}+\mathscr{P})}{\mathrm{d}t} = -\mathscr{D}. \tag{5.10}$$

Equations (5.7) and (5.10) were derived by D. D. Joseph.

In §6 we shall show that rigid motions are globally stable, as is the case with one fluid, but that the stable configurations of the rigid motions minimize $\mathscr{P}$ subject to the volume constraint (2.6).

It is useful to write the potential in a dimensionless form in which $R = d+\delta$ and $(\delta, x, 1/\alpha)$ are made dimensionless with d and $\Delta = \delta/d$. Then, to within constants, we may define a dimensionless potential

$$\tilde{\mathscr{P}} = -\langle\!\langle 1\rangle\!\rangle + \langle\!\langle [(1+\Delta)^2(1+\Delta_x^2)+\Delta_\theta^2]^{\frac{1}{2}}\rangle\!\rangle + \tfrac{1}{8}J\langle\!\langle [2\Delta+\Delta^2]^2\rangle\!\rangle, \tag{5.11}$$

where
$$J = -\frac{d^3[\![\rho]\!]\,\Omega^2}{T}.$$

The constraint (2.6) implies that

$$\langle\!\langle 2\Delta+\Delta^2\rangle\!\rangle = 0. \tag{5.12}$$

This shows that the average deviation $\Delta = -1+R/d$ from zero must be negative if the volume of the two fluids is preserved.

It is necessary to remark that the representation $r = R(\theta, x, t)$ of the free surface is not completely general and it loses its utility when the magnitude $|\Delta|$ of the deviation from the cylinder is equal to

$$\tilde{\Delta} = \min\left[1-\frac{a}{d}, \frac{b}{d}-1\right]. \tag{5.13}$$

If $|\Delta| \geqslant \tilde{\Delta}$, then the interface will touch one or the other of the cylinders, and a smooth free surface will not be possible.

The linearized form of (5.11) for Δ near to zero is

$$\tilde{\mathscr{P}} = \tfrac{1}{2}\langle\!\langle [J-1]\,\Delta^2+\Delta_x^2+\Delta_\theta^2\rangle\!\rangle. \tag{5.14}$$

6. Stability of rigid motion

The following results concern solutions of the equations that are smooth for all time.

THEOREM 1. *Rigid motions are stable in the sense that periodic disturbance of rigid motion must decay in the mean.*

We first note that $\mathscr{D} > 0$ on all non-zero disturbances of rigid motion. We are considering the stability of flows with heavy fluid outside, $-[\![\rho]\!] = \rho_2 - \rho_1 > 0$, or inside, $[\![\rho]\!] > 0$. These two situations, called (*A*) and (*B*) in §2, are distinct in that they cannot be connected by time-dependent motions with smooth interfaces. We choose $\boldsymbol{u}$ to be a disturbance of one or the other of these two possibilities. Then the positivity of $\mathscr{D}$ implies the decay of $\mathscr{E}+\mathscr{P}$. In fact, we could show that there is $\lambda > 0$ depending on the viscosities and the densities, such that $\mathscr{D} \geqslant \lambda\mathscr{E}$, where λ depends on $\mathscr{V}_1$, $\mathscr{V}_2$, ρ_1, ρ_2, μ_1 and μ_2. Integrating (5.10) from $t = 0$ to t, we find that

$$\mathscr{E}(t)+\mathscr{P}(t) = \mathscr{E}(0)+\mathscr{P}(0)-\int_0^t \mathscr{D}(\tau)\,\mathrm{d}\tau$$

$$\leqslant \mathscr{E}_0+\mathscr{P}_0-\lambda\int_0^t \mathscr{E}(\tau)\,\mathrm{d}\tau; \tag{6.1}$$

it follows that

$$\lambda\int_0^t \mathscr{E}(\tau)\,\mathrm{d}\tau \leqslant \mathscr{E}(0)+\mathscr{P}(0)-\mathscr{E}(t)-\mathscr{P}(t). \tag{6.2}$$

Since $\mathscr{P}$ is bounded from below we conclude that

$$\mathscr{E}(t) \text{ and } \mathscr{D}(t) \text{ are integrable.} \tag{6.3}$$

Moreover, assuming that $\mathscr{E}$ goes to 0 as $t \to +\infty$, $\mathscr{P}$ admits a finite limit as t goes to $+\infty$.

Let us consider the limit configuration $(\mathscr{E}(\infty), \mathscr{P}(\infty))$; since $\mathscr{E}(\infty) = 0$, this is a rigid motion. To show that $\mathscr{P}(\infty)$ is a minimum of the functional $\mathscr{P}$ as R varies, we consider any rigid motion $(\mathscr{E}(0) = 0, \mathscr{P}(0))$ and assume that this rigid motion goes to the rigid motion $(\mathscr{E}(\infty) = 0, \mathscr{P}(\infty))$ as t goes to $+\infty$. If $\mathscr{P}(0) \neq \mathscr{P}(\infty)$, then the interface between the two liquids must have moved from the configuration at $t = 0$ to the one at $t = \infty$, and in this motion $\mathscr{D}(t) > 0$ on an interval with non-zero measure. Then, from (5.10),

$$\mathscr{P}(\infty)-\mathscr{P}(0) = -\int_0^\infty \mathscr{D}(t)\,\mathrm{d}t < 0, \tag{6.4}$$

so that

$$\mathscr{P}(\infty) < \mathscr{P}(0). \tag{6.5}$$

Thus $\mathscr{P}$ decreases in every change of configuration between rigid motions. Since $\mathscr{P}$ is a bounded-from-below functional of R, $\mathscr{P}$ must decrease to

$$\mathscr{P}(\infty) = \lim_{t\to\infty}\mathscr{P}(t) = \min_R \mathscr{P}(R), \tag{6.6}$$

where $\overline{R}^2 = d^2$ and R is periodic and continuously differentiable in x and θ.

We may describe the set of such periodic functions R with $\overline{R}^2 = d^2$ as a one-parameter family $R(\epsilon)$. We assume that $R(0) = \tilde{R}$, where

$$\mathscr{P}(\tilde{R}) = \min_R \mathscr{P}(R) = \min_\epsilon \mathscr{P}(R(\epsilon)). \tag{6.7}$$

We must have:

$$\frac{d\mathscr{P}}{d\epsilon}[R(\epsilon)]_{\epsilon=0} = -\langle\!\langle \tilde{R}\frac{dR}{d\epsilon}(0)\,(2\tilde{H}T+\tfrac{1}{2}[\![\rho]\!]\,\Omega^2\tilde{R}^2)\rangle\!\rangle = 0, \tag{6.8}$$

where $2\tilde{H}$ denotes the curvature (2.13) evaluated on the minimizer $R = \tilde{R}$. Since $\overline{R^2}(\epsilon) = d^2$, we have

$$\langle\!\langle \tilde{R}\frac{dR}{d\epsilon}(0)\rangle\!\rangle = 0, \tag{6.9}$$

so that $(\tilde{R}\,dR/d\epsilon)(0)$ is orthogonal to constants. It follows now from (6.8) and (6.9) that

$$2\tilde{H}T+\tfrac{1}{2}[\![\rho]\!]\,\Omega^2\tilde{R}^2 = C = 2\bar{\tilde{H}}T+\tfrac{1}{2}[\![\rho]\!]\,\Omega^2 d^2, \tag{6.10}$$

where $\bar{\tilde{H}}$ is the mean value of $\tilde{H}$.

Equation (6.10) may be recognized as the differential equation arising from the normal-stress condition (2.12) on rigid motions.

It now follows that extended rigid motions, which are globally stable, are actually hydrodynamically admissible, with a balanced normal-stress equation, when the interface $R = \tilde{R}$ is a minimizer of (6.6). The stable rigid motions satisfy (2.5) in the form

$$0 = \frac{\partial\tilde{R}}{\partial t}+\Omega\frac{\partial\tilde{R}}{\partial\theta},$$

where $\tilde{R}$ is time-independent in a rotating coordinate system.

THEOREM 2. *The stable configurations are those rigid motions that minimize $\mathscr{P}$ among $C'(x,\theta)$ functions $R(x,\theta)$ satisfying the volume constraint* (2.6).

Consider first case (A) in which the heavy fluid is outside, $J > 0$. If $J > 1$, then $\bar{\mathscr{P}}$ given by (5.14) is a minimum when $\Delta = 0$.

THEOREM 3. *The cylindrical interface with constant radius $R = d$ is stable against small disturbances if and only if $J \geqslant 1$.*

It is of interest to ask when $R = d$ is a global minimum of $\mathscr{P}$ among all interfaces of the form $r = R(\theta, z)$ compatible with the volume constraint. This question is answered by the following theorem of M. Renardy.

THEOREM 4. *The concentric interface $R = d$ is a global minimum of $\mathscr{P}$ among all interfaces $r = R(\theta, z)$ satisfying* (2.6) *if and only if $J \geqslant 4[1+a/d]^{-2}$, $0 \leqslant a \leqslant d$.*

First we show that the criterion of theorem 4 is sufficient for stability. Certainly we have $Q \geqslant \hat{Q} = \langle\!\langle TR+\tfrac{1}{8}\Omega^2(\rho_2-\rho_1)\,R^4\rangle\!\rangle$, and it is sufficient to show that $R = d$ minimizes $\hat{Q}$. We set $R^2 = d^2(1+\gamma)$, hence γ is subject to the constraints $-1+a^2/d^2 \leqslant \gamma \leqslant -1+b^2/d^2$ and $\langle\!\langle\gamma\rangle\!\rangle = 0$. We then have

$$\begin{aligned}\hat{Q} &= Td\langle\!\langle(1+\gamma)^{\frac{1}{2}}+\tfrac{1}{8}J(1+\gamma)^2\rangle\!\rangle\\ &= Td\langle\!\langle(1+\gamma)^{\frac{1}{2}}+\tfrac{1}{8}J(1+\gamma)^2-\gamma[\tfrac{1}{2}+\tfrac{1}{4}J]\rangle\!\rangle.\end{aligned} \tag{6.11}$$

We define $$f(\gamma) = (1+\gamma)^{\frac{1}{2}}+\tfrac{1}{8}J(1+\gamma)^2-\gamma[\tfrac{1}{2}+\tfrac{1}{4}J]. \tag{6.12}$$

The constant multiplying γ has been chosen such that $f'(0) = 0$. If $J \geqslant 4[1+a/d]^{-2}$, then f has its minimum at $\gamma = 0$, for γ in the range

$$[-1+a^2/d^2,\ -1+b^2/d^2].$$

The criterion is also necessary for a global minimum. We can choose R such that $R_\theta = 0$, and, by choosing a small (long waves) we can make R_z as small as we like.

It follows that it is also necessary that $R = d$ minimizes $\hat{Q}$. If $J < 4[1+a/d]^{-2}$, then $f[-1+a^2/d^2] < f(0)$, and the graph of f is sketched as figure 1. If we draw the tangent from the point $(-1+a^2/d^2, f(-1+a^2/d^2))$ as indicated, it will touch the graph of f at a point $(\tilde{\alpha}, f(\tilde{\alpha}))$ to the right of $\gamma = 0$. Let $\hat{\alpha}$ be any number such that $0 \leqslant \tilde{\alpha} \leqslant \max(\tilde{\alpha}, -1+a^2/d^2)$. The straight line connecting the points $(-1+a^2/d^2, f(-1+a^2/d^2))$ and $(\hat{\alpha}, f(\hat{\alpha}))$ intersects the line $\gamma = 0$ at a value below $f(0)$. Let us now consider a perturbation $\gamma(x)$ with the following properties:

(i) γ takes only the values $\hat{\alpha}$ and $-1+\dfrac{a^2}{d^2}$,

(ii) $\langle\!\langle \gamma \rangle\!\rangle = 0$.

Let p and $q = 1 = p$ be the probabilities with which γ takes the values $\hat{\alpha}$ and $-1+a^2/d^2$ respectively. Then

$$\frac{\alpha}{4\pi^2}\langle\!\langle \gamma \rangle\!\rangle = p\hat{\alpha} + q\left(-1+\frac{a^2}{d^2}\right) = 0, \quad \frac{\alpha}{4\pi^2}\langle\!\langle f(\gamma) \rangle\!\rangle = pf(\hat{\alpha}) + qf\left(-1+\frac{a^2}{d^2}\right).$$

Let $y(\gamma)$ be a point on the tangent line of figure 1. After eliminating p from the last two equations, we find that

$$\frac{\alpha}{4\pi^2}\langle\!\langle f(\gamma) \rangle\!\rangle = \frac{\left(-1+\dfrac{a^2}{d^2}\right)f(\hat{\alpha}) + \hat{\alpha}f\left(-1+\dfrac{a^2}{d^2}\right)}{-1+\dfrac{a^2}{d^2}+\hat{\alpha}}$$

$$= y(0) < f(0).$$

Hence $\gamma = 0$ does not minimize $\hat{Q}$.

Turning next to case (B) in which the heavy fluid is inside, $J < 0$, we find that $f(\gamma)$ is concave. Hence $\hat{Q}$ will have minimizers only at boundary values $-1+a^2/d^2$ and $-1+b^2/d^2$. In an infinite cylinder there will therefore be no minimizers of P of the form $r = R(\theta, z)$. In a finite cylinder, we cannot make R_z arbitrarily small without also making γ small, and there may be stable motions with heavy fluid inside, which have a corrugated free surface, as in the experiments of Yih (1960) and Moffatt (1977). It would be of interest to determine these corrugated shapes as a solution for the minimum problem of Q. If J is large, we expect the amplitude of the corrugated surface to be also large and eventually violate the constraints $-1+a^2/d^2 < \gamma < -1+b^2/d^2$.

The results of this paper have some relevance for the problem of centrifuging. Intuitively one expects that the heavy fluid will be outside if the rate Ω of rotation is large, even if the heavy fluid were initially on the inner cylinder. The transport of fluid from the inner to the outer cylinder is a topologically complex process which cannot be handled in the frame of our smooth parametrization of the interface. The transpot of fluid from the inner to the outer surface is also a physically complex process involving the rupture of adhesion at both walls and possibly internal fracturing and healing of the liquids themselves. These physical processes are not well understood, and they do not appear in our equations. Nevertheless, it is not unreasonable to seek stable configurations among those that minimize $\mathscr{P}$ with respect also to the position of heavy and light fluid.

We can compare the potentials $\mathscr{P}_A$ and $\mathscr{P}_B$ for flow with heavy fluid inside, $[\![\rho]\!] > 0$, and heavy fluid outside, $[\![\rho]\!] < 0$, under the condition that the volume m, defined

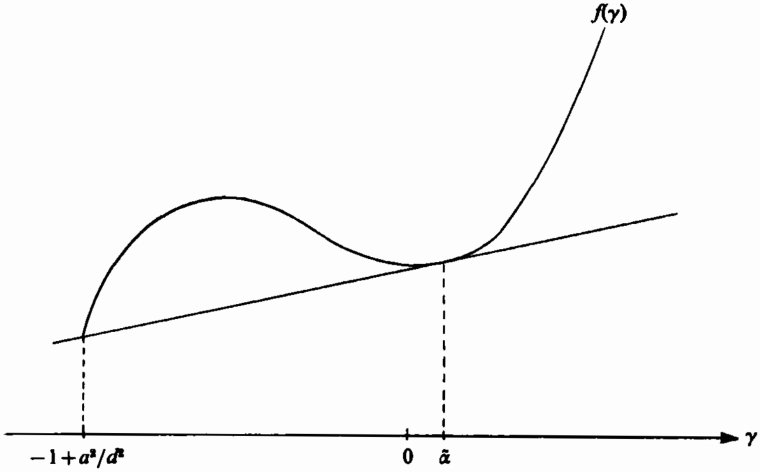


FIGURE 1

in §2, is fixed. The total kinetic energy $\mathscr{K}_A$ when the heavy fluid with density ρ_1 is inside is given by

$$\mathscr{K}_A = \mathscr{K}_1+\mathscr{K}_2 = \tfrac{1}{8}\rho_1\,\Omega^2 (\!(R_A^4-a^4)\!) + \tfrac{1}{8}\rho_2\,\Omega^2 (\!(b^4-R_A^4)\!),$$

where $(\!(R_A^2)\!) = (\!(d_A^2)\!)$. When the heavy fluid with density ρ_1 is outside,

$$\mathscr{K}_B = \mathscr{K}_1+\mathscr{K}_2 = \tfrac{1}{8}\rho_1\,\Omega^2 (\!(b^4-R_B^4)\!) + \tfrac{1}{2}\rho_2\,\Omega^2 (\!(R_B^4-a^4)\!),$$

where $(\!(R_B^2)\!) = (\!(d_B^2)\!)$. Using (2.10), with $m_A = m_B = m$, we find that

$$(\!(R_A^2)\!) = \frac{mb^2+a^2}{1+m}, \quad (\!(R_B^2)\!) = \frac{b^2+ma}{1+m}.$$

The potentials are $\quad \mathscr{P}_A = -\mathscr{K}_A + T|\Sigma_A|, \quad \mathscr{P}_B = -\mathscr{K}_B + T|\Sigma_B|,$

where
$$|\Sigma_A| = \int_{\Sigma_A} \mathrm{d}\Sigma$$

is the area of Σ_A. The difference in the potentials is

$$\begin{aligned}\mathscr{P}_A - \mathscr{P}_B &= -\mathscr{K}_A + \mathscr{K}_B + T(|\Sigma_A| - |\Sigma_B|) \\ &= \tfrac{1}{8}[\![\rho]\!]\,\Omega^2 (\!(b^4+a^4-R_A^4-\mathrm{R}_B^4)\!) + T(|\Sigma_A|-|\Sigma_B|).\end{aligned}$$

Here $[\![\rho]\!] = \rho_1-\rho_2 > 0$. If $\mathscr{P}_A > \mathscr{P}_B$ then (B) with heavy fluid outside is more stable. Since $(\!(R^2)\!) = (\!(d^2)\!)$, we have

$$(\!([R^2-d^2]^2)\!) = (\!(R^4)\!) - (\!(d^4)\!).$$

The potential difference may be written as

$$\mathscr{P}_A - \mathscr{P}_B = \tfrac{1}{8}[\![\rho]\!]\,\Omega^2 [(\!(b^4+a^4-d_A^4-d_B^4)\!) - (\!([R_A^2-d_A^2]^2 + [R_B^2-d_B^2]^2)\!)] + T(|\Sigma_A|-|\Sigma_B|).$$

Now, using (2.11), we eliminate d_A^4 and d_B^4:

$$\mathscr{P}_A - \mathscr{P}_B = \tfrac{1}{8}[\![\rho]\!]\,\Omega^2 \left[\frac{2m(\!([b^2-a^2]^2)\!)}{(1+m)^2} - (\!([R_A^2-d_A^2]^2 + [R_B^2-d_B^2]^2)\!)\right] + T|\Sigma_A| - T|\Sigma_B|.$$

Consider now the case of uncorrugated interfaces with $R_A = d_A$, $R_B = d_B$, $\Sigma_A = 2\pi d_A$, $\Sigma_B = 2\pi d_B$. We find that

$$\mathscr{P}_A - \mathscr{P}_B = \tfrac{1}{4}[\![\rho]\!]\,\Omega^2 m\,\frac{\langle\!\langle [b^2-a^2]^2 \rangle\!\rangle}{(1+m)^2}\,2\pi T \langle\!\langle d_A - d_B \rangle\!\rangle,$$

where d_A and d_B are given in terms of m by (2.11). If the volume ratio m of heavy (ρ_1) to light (ρ_2) fluid is greater than one, then $d_A > d_B$. Hence $\mathscr{P}_A > \mathscr{P}_B$ when $m > 1$. If there is only a small amount of heavy fluid, $m < 1$ and $d_B < d_A$, then $\mathscr{P}_A > \mathscr{P}_B$ if $[\![\rho]\!]\,\Omega^2/T$ is large. In all these cases the configuration with heavy fluid outside is more stable. If $[\![\rho]\!]\,\Omega^2/T$ is small enough, then $\mathscr{P}_A < \mathscr{P}_B$, and the configuration with heavy fluid inside is more stable. However, our earlier analysis showed that, when the heavy fluid is inside, the $R_B(\theta, x)$ that minimizes $\mathscr{P}$ is not everywhere equal to d_B.

7. Stability of rollers

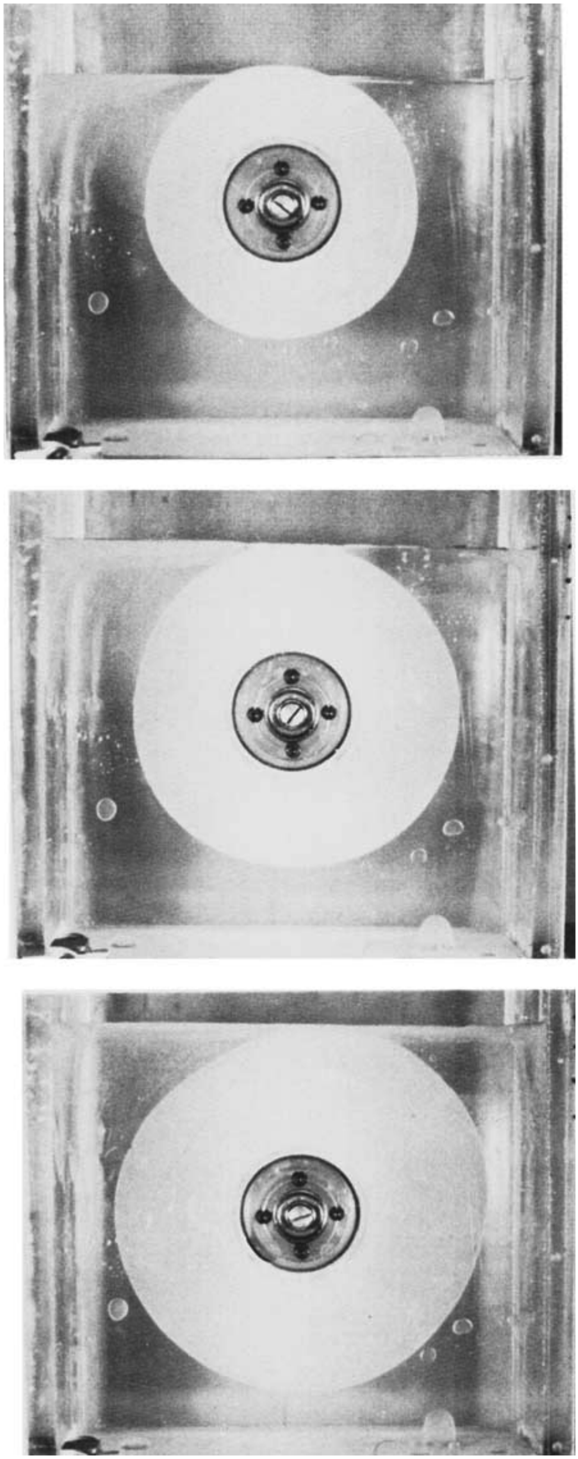
Rollers are viscous fluid bodies that rotate as rigid wheels in fluids of smaller viscosity. These rollers have been observed (Joseph *et al.* 1984) in bicomponent flows of immiscible liquids in several different flow configurations: on a cylinder rotating in a box, between the four cylinders of Taylor's mill for studying straining flows, and separating dynamically driven Taylor vortices between rotating cylinders.

The most interesting feature of the dynamics leading to the formation of rollers is the fracturing of the viscous liquid at some critical level of the stress. In this process the roller breaks away from the sidewall and relieves the high stress associated with no slip at the sidewall. So in the final, stable dynamics, rollers are lubricated by water and air on all sides. The rollers rotate nearly as rigid bodies because they are so viscous. The stability of rollers, as our analysis suggests, depends on the fact that the density stratification is such as to prevent the centrifuging of the roller. The viscosity ratio is probably not an important factor in the dynamics of stable rollers.

The water that surrounds the rollers in experiments is at rest near the tank wall and cannot rotate rigidly. Therefore rollers are not a special case of rigid motions studied in this paper. However, the density stratification, with water outside, does contribute to the stability of rollers, with a stabilizing term $[\![\rho V^2]\!]\,\delta^2$ at the interface, where V is the common velocity of fluid particles on either side of the interface and δ is the surface deflection; here assumed small.

Rollers are unstable to non-axisymmetric disturbances when the angular velocity is high enough. This instability is associated with viscous shearing, which becomes important at higher speeds and with a possible unstable distribution of angular momentum.

The low-speed rollers are robustly stable. In our analysis we did not consider gravity, but gravity does enter into the dynamics of the stable rollers reported in Joseph *et al.* (1984) and here. In experiments in which the top of the roller rotates in air the roller would centrifuge out into the air were it not for gravity, which on the small top portion of the roller exposed to air is nearly radial. A similar, but smaller, effect due to gravity occurs at the bottom of the roller, which is pushed up by gravity because the lighter oil is buoyant in water. Gravity tends to flatten rollers into right-circular cylinders. To a degree the diameter of stable rollers can be controlled by gravity, with a tendency for the roller to poke its head into the air. We are able to change the diameter of the rollers by changing the water level in the box. This effect of gravity is exhibited in figures 2 (*a–c*). Sketches of the side view of these plates are shown in figures 2 (*d–f*).



(a)

(b)

(c)

FIGURE 2 (a), (b) and (c). For caption see facing page.

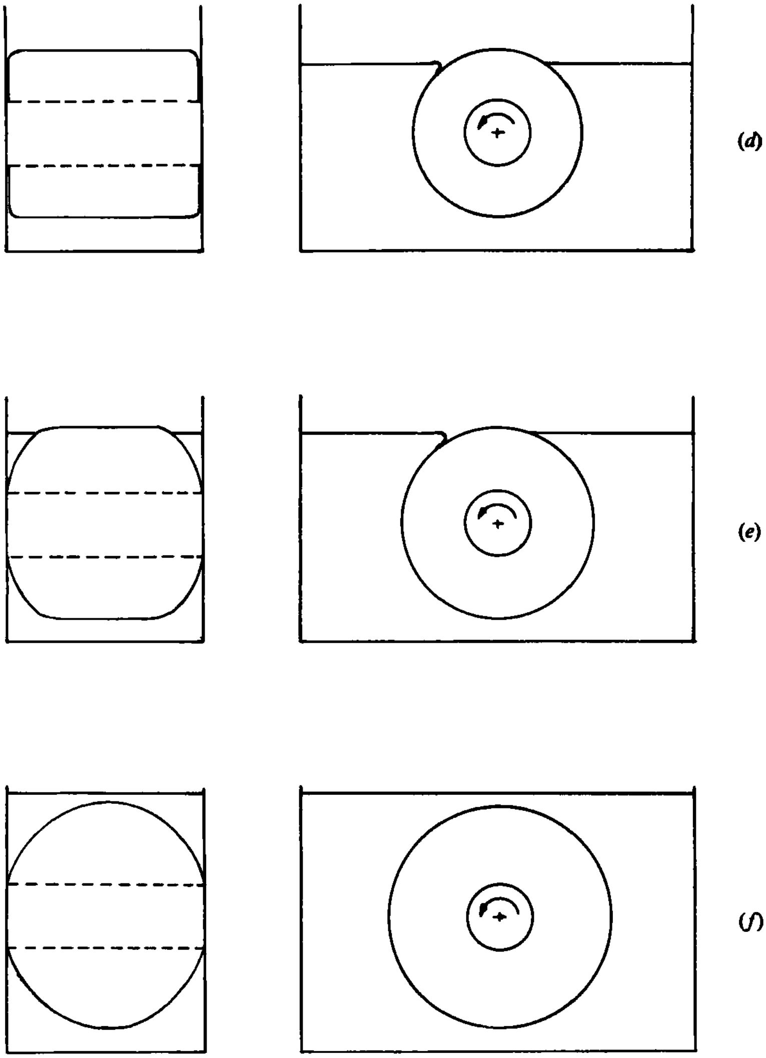
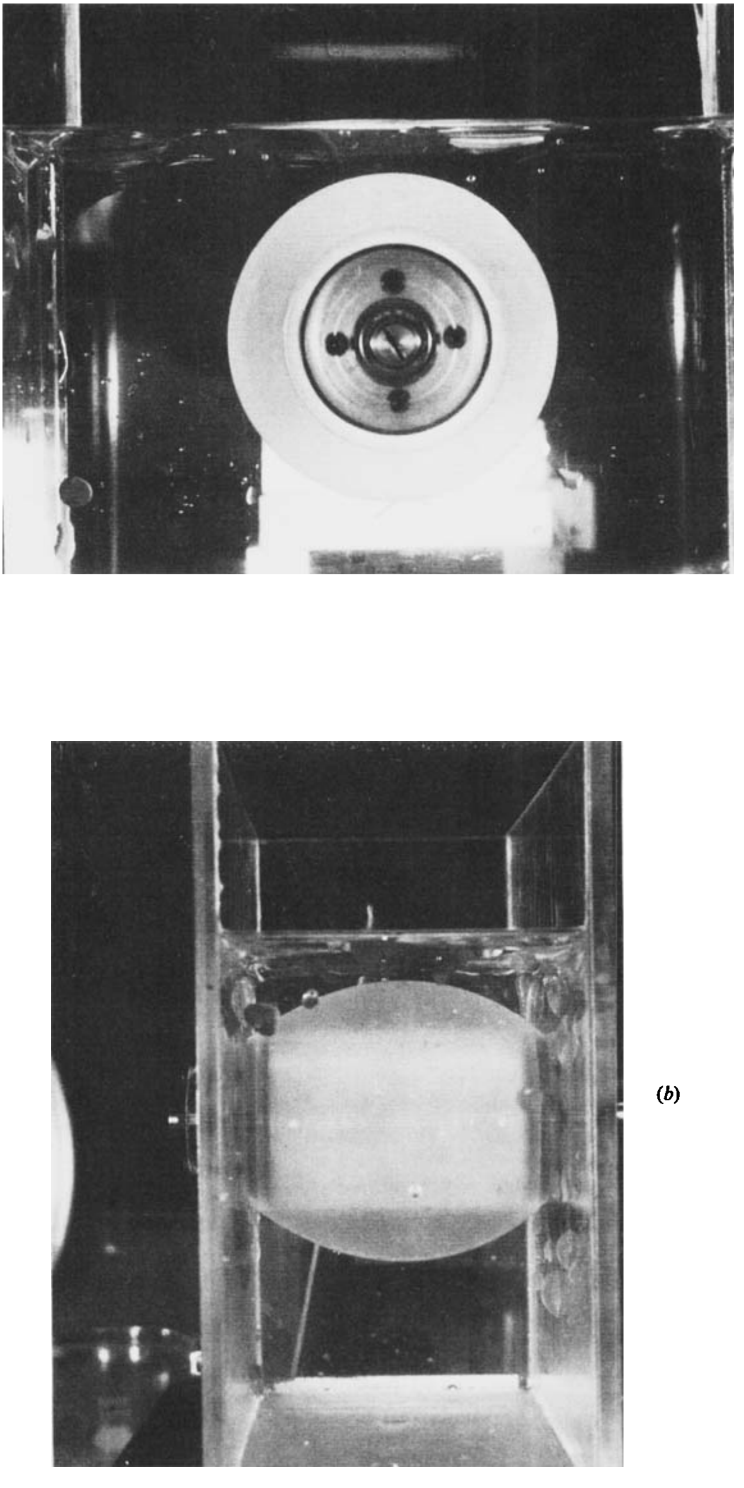


FIGURE 2. Roller of silicone oil ($\rho = 0.95$ g/cm^3, $\mu = 95000$ cP) in water at different water levels. The rod is made of Plexiglas, 2 in. in diameter, and rotates at 10 r.p.m. (*a*) The roller is very nearly in a solid-body rotation with small shearing by water at the roller rim. Part of the roller is in water and the other in air. The roller is very stable, held together by hydrostatic pressure in water and gravity in air. (*b*) Water is added to the box. The roller becomes larger by flattening out but remains round and stable. (*c*) More water is added. The roller becomes even larger. The roller is now completely submerged in water and is slightly out of round due to buoyancy. (*d*) Sketch of the side view and front view of (*a*). (*e*) Sketch of the side view and front view of (*b*). Water is added to the box. The diameter of the roller becomes larger. The shape of the roller changes, conserving volume. (*f*) Sketch corresponding to (*c*).

The principal effect of gravity may be eliminated by submerging the roller entirely in water, as in figures 3(*a*, *b*). When the flattening effects of gravity are absent, the shape of the interface on stable rollers is strongly influenced by interfacial tension, with bounding surfaces in nearly circular arcs, as in figure 3(*b*). The pressure distribution in the water is not a strong barrier to centrifuging, and the dynamics of the roller in figure 3 are closer to case (*B*) of this paper in which the heavy fluid is on the inside cylinder with a corrugated interface separating the two liquids.



(a)

(b)

FIGURE 3. (a) Front view of a completely submerged roller rotating at about 1.5 r.p.m. (b) Side view of the submerged roller of (a).

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